

Lorentz contraction and accelerated systems

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Abstract. The paper discusses the problem of the Lorentz contraction in accelerated systems, in the context of the special theory of relativity. Equal proper accelerations along different world lines are considered, showing the differences arising when the world lines correspond to physically connected or disconnected objects. In all cases the special theory of relativity proves to be completely self-consistent.

1. Introduction

The behavior of accelerated systems in special relativity is a delicate problem, that deserves a careful analysis lest apparent paradoxes and contradictions seem to come up. Usually more attention is paid in textbooks to kinematical 'paradoxes', whilst accelerated systems are not discussed at length. This at least is what emerges from reading such classic texts as [1], [2], [3], [4], [5], [6], [7], [8], [9].

Of course acceleration plus the equivalence principle is a fundamental and delicate issue, considering that gravitation and acceleration are locally indistinguishable. We think that accelerated systems should be fully discussed in order to obtain a good understanding of special relativity.

In this paper a simple problem will be considered which, at a superficial glance, can disorient students. The final conclusion will be that special relativity, once again, is self consistent.

2. Posing the problem

Two objects at different positions along the x axis of an inertial reference frame undergo equal accelerations during equal coordinate time intervals. In this condition we expect that the distance between them, seen in the initial rest frame, always remains equal to its initial (rest) value. At the end of the accelerated phase however, when both bodies will move with the same constant speed, the distance between them should turn out to be Lorentz contracted with respect to the rest (proper) value. Is there a contradiction? Has the proper distance increased during the acceleration time? If the two objects are physically connected, has a stress set in?

2.1. One observer, one meter

Suppose, to begin with, that there is a fixed inertial observer at the origin; let us call him O . Then add a second observer O' , initially coincident with O ; both observers carry equal meter rods, stretched along the x axis; the length of the rods in the rest inertial frame is l . O' is set into accelerated motion along the x axis starting at $t = 0$; let us assume, for the sake of simplicity, that the proper acceleration of O' is a constant a .

The world line of O' in the reference frame of O is a hyperbola described by the equation [10]

$$\left(x + \frac{c^2}{a}\right)^2 - c^2 t^2 = \frac{c^4}{a^2} \quad (1)$$

The metric in the frame of O is of course (letting aside the irrelevant y and z dimensions):

$$ds^2 = c^2 dt^2 - dx^2$$

From the view point of O' the acceleration produces the same effect as a uniform gravitational field along x^\ddagger , consequently a rod aligned with x will be compressed and accordingly shortened with respect to a rod along, say, y . The rigid rod case has been discussed in [13]. In our case no rigidity is assumed for the extended rods since it would contradict the relativity theory. The amount of the deformation of the rod along x will depend on the nature of the rod and, specifically, on the stiffness of the material which it is made of. Calling k the stiffness of the rod and λ the proper density (per unit length) of its material, then l' , which is the length of the rod as seen by O' , turns out to be§

$$l' = l \frac{k}{k + \lambda a} \quad (2)$$

The length seen by O is obtained from l' projecting it from the x' axis (the space of O') to the x axis, i.e. multiplying by $\sqrt{1 - v^2/c^2}$, according to the standard Lorentz contraction. The instantaneous coordinate velocity $v = \frac{dx}{dt}$, as seen by O , is calculated from (1):

$$v = \frac{ct}{\sqrt{\left(\frac{c^2}{a^2} + t^2\right)}} = \frac{at}{\sqrt{1 + \frac{a^2 t^2}{c^2}}} \quad (3)$$

Finally the length seen by O will be

$$l'' = l' \sqrt{1 - \frac{v^2}{c^2}} = \frac{l'}{\sqrt{\left(1 + \frac{t^2 a^2}{c^2}\right)}} = \frac{k}{k + \lambda a} \frac{l}{\sqrt{\left(1 + \frac{t^2 a^2}{c^2}\right)}}$$

The rod, in the frame of O , gets contracted more and more as time passes. Considering this fact, since the effect depends on the properties of material bodies, all lengths measured along the x direction using the rod appear distorted with respect to the initial rest values. Nothing happens to the distances measured in any transverse direction.

When the acceleration phase comes to an end at coordinate time t_0 , the translational velocity, with respect to O , keeps its constant final value

$$v_0 = \frac{ct_0}{\sqrt{\left(\frac{c^2}{a^2} + t_0^2\right)}}$$

‡ The issue of the local equivalence between accelerated systems and gravitational fields is not a trivial one (see for instance [11] and [12]). In our case however the situation is simple if we assume that a positive acceleration is applied to the rear end of the rod, which is otherwise free. The possibility of using an infinite set of inertial frames to describe an accelerated observer deserves in turn a careful discussion. In our case however all these problems are not relevant for the final conclusions.

§ The rod, in the uniformly accelerated frame, is in equilibrium under the action of the elastic force and the gravitational-like force along x , due to its acceleration. Hence, the total potential energy can be written as

$$W = \frac{1}{2}k(l' - l)^2 + \frac{1}{2}\lambda a l'^2$$

where λ is assumed to be a constant. Differentiating with respect to l' , formula (2) is obtained.

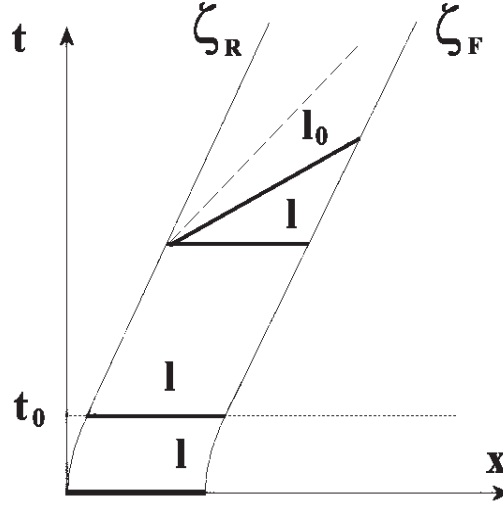


Figure 1. The world lines ζ_R and ζ_F of two objects accelerated with the same proper acceleration and for the same proper times up to $t = t_0$ are shown. Afterwards both objects continue to move inertially at the same constant speed. The distance between the objects as seen by a static observer maintains its initial value l , but the proper distance increases to become l_0 . The length l corresponds to the Lorentz contracted l_0 . The dashed line represents the light cone.

and a drops to zero in (2). Not considering oscillations and internal energy dissipation, the rod recovers its initial proper length, and the corresponding length seen by O is

$$l''_0 = l \sqrt{1 - v_0^2/c^2}$$

The result is exactly what was expected to be: no paradox of any sort appears, all measured lengths recover the initial unaccelerated values.

3. Light rays

Let us now consider a situation where two equal rockets are initially placed on the x axis at a distance l from one another. Every rocket carries on board a scientist to make measurements, and an engineer to control the thrust of the rocket. The engineers carry identical (initially) synchronized clocks and have the same instructions for the regime of the engines. Let us call F the front rocket (and moving observer), and R the rear rocket, with its observer (see figure 1). F and R are not physically connected, so that they move exactly with the same proper acceleration at any time. The way used to monitor the reciprocal positions is the exchange of light rays.

The infinitesimal proper time interval $d\tau$ is given, in terms of the coordinate time interval dt , by

$$d\tau = dt \sqrt{1 - \frac{v^2}{c^2}} \equiv \gamma^{-1} dt \quad (4)$$

where we have introduced the Lorentz factor $\gamma(v) = 1/\sqrt{1 - \frac{v^2}{c^2}}$. Hence, substituting v from 3, we obtain

$$d\tau = \frac{dt}{\sqrt{1 + \frac{a^2 t^2}{c^2}}} \quad (5)$$

in terms of the coordinate time. By integrating ($\tau = 0$ when $t = 0$), the proper time lapse turns out to be

$$\tau = \frac{c}{a} \sinh^{-1} \left(\frac{a}{c} t \right) \quad (6)$$

hence, we obtain

$$t = \frac{c}{a} \sinh \left(\frac{a}{c} \tau \right) \quad (7)$$

Now, if we substitute in 3 this expression of coordinate time, as a function of the elapsed proper time, we see that the coordinate velocity can be written as

$$v = c \tanh \left(\frac{a}{c} \tau \right) \quad (8)$$

At a given predetermined τ_0 the engines are stopped on both rockets. From that moment on, both for R and F , the flight continues at a constant coordinate speed

$$v_0 = \frac{dx}{dt} = c \tanh \left(\frac{a}{c} \tau_0 \right) \quad (9)$$

and the corresponding Lorentz factor $\gamma(v_0)$ is

$$\gamma(v_0) = \cosh \left(\frac{a}{c} \tau_0 \right) \quad (10)$$

The round trip of light ([2]) between the space ships corresponds to a coordinate time interval

$$c\delta t = 2l \cosh^2 \left(\frac{a}{c} \tau_0 \right)$$

and in terms of proper time of R ($\delta t = \gamma(v_0)\delta\tau$)

$$c\delta\tau = 2l \cosh \left(\frac{a}{c} \tau_0 \right) \quad (11)$$

The proper distance is usually defined as $l_0 = \frac{c\delta\tau}{2}$. On the other hand, the distance seen by a static observer in these circumstances remains always equal to l , but the proper distance in the frame of the rockets, l_0 , has progressively increased during the acceleration so that its Lorentz contraction ($l = \gamma^{-1}l_0$) produces precisely the l result. In fact, from (11) one has

$$l_0 \equiv \frac{c\delta\tau}{2} = l \cosh \left(\frac{a}{c} \tau_0 \right) \quad (12)$$

which, considering (9) or (10), corresponds to

$$l_0 = \frac{l}{\sqrt{1 - \frac{v_0^2}{c^2}}} \quad (13)$$

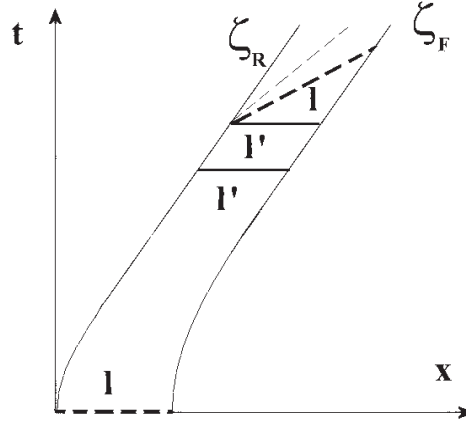


Figure 2. ζ_R and ζ_F represent the world lines of two rockets connected by a spring, as explained in the text. Once the acceleration finishes (at different coordinate times for the two rockets) the ends of the spring will keep on moving at a constant coordinate speed, and the spring will recover its initial unstretched length l . A static observer will measure the Lorentz contracted length l' .

4. Two physically connected accelerated observers

Let us now consider a situation where both observers at R and F are physically connected by a spring.

In this condition, one could expect the two ends of the rod to be equally accelerated, however, just as in a gravitational field, when trying to keep the length of a spring fixed notwithstanding gravity, the spring will react with a pull on both ends, since its rest length is now shorter than what it would be without acceleration. The consequence will be that the actual acceleration of the front end will be a little bit less than what the engine alone would produce, and the acceleration of the rear end will be a little bit more for the same reason. In this way, the proper times of the two engineers will no longer be the same at a given coordinate time and the two world lines of the ends of the spring will no longer be equal hyperbolae (see figure 2). In fact, the rear world line will in general be more curved than the front one. When the engines stop thrusting, at the same proper times of the engineers, but at different coordinate times, after some transient (including oscillations and dissipation of energy) the situation will be such that the spring will remain unstretched in its rest frame, i.e. its proper length will again be l and of course its both ends will move at the same coordinate speed. The Earth bound observer will measure a properly contracted length

$$l' = l \sqrt{1 - \frac{v^2}{c^2}}$$

5. Conclusion

As we have seen by elementary considerations, the simple scheme of two equally accelerated observers leads to results consistent with the Lorentz contraction in any case. When the two observers are physically connected, the material bridge between them insures the classical contraction. When the two observers are not connected, the relativity of simultaneity produces, from the view point of the rear observer, a forward flight of the front observer, which in the end will be seen as an increase in the proper distance, leading, via Lorentz contraction, to a distance, in the frame of the static observer, exactly equal to the initial one.

There is no need to analyze in detail the accelerated phase, when such concepts as an extended proper distance are ill defined. Actually the same conclusions are reached no matters what the acceleration programs are, provided they are equal with respect to proper times. The reason why we have considered constant proper acceleration has been just for the sake of simplicity in intermediate steps. We have also implicitly assumed that the sizes of physical systems we considered, were small enough not to incur into troubles with horizons and other difficulties typical of extended accelerated reference frames [10][14]

We think that proposing an example/exercise of this sort to the students would produce a deeper insight in the principles of special relativity.

References

- [1] Rindler W., *Special Relativity*, Oliver and Boyd, Edinburgh and London, 1960
- [2] Rindler W., *Essential Relativity*, Van Nostrand Reinhold Company, New York, 1969
- [3] Landau L.D. and Lifshitz E.M., *The Classical Theory of Fields*, Pergamon Press, New York, 1971
- [4] Ohanian H.C. and Ruffini R., *Gravitation and Spacetime*, W.W. Norton and Company, 1994
- [5] Bohm D., *The Special Theory of Relativity*, Addison Wesley Pub. Company, New York, 1989
- [6] Rosser W.G.V., *An introduction to the Theory of Relativity*, Butterworths, London, 1964
- [7] J.G. Taylor, *Special Relativity*, Oxford Physics Series, Oxford, 1975
- [8] R. Katz, *An introduction to the Special Theory of Relativity*, D. Van Nostrand Company Inc., New York, 1964
- [9] Adler R, Bazin M., Schiffer M., *Introduction to General Relativity*, McGraw-Hill, New York, 1965
- [10] Misner C.W., Thorne K.S., Wheeler J.A., *Gravitation*, chapt. 6, Freeman Ed., S.Francisco, 1973
- [11] Desloge E. A., (1989), *Amer. Jour. Phys.*, **57**, 1121-1125
- [12] Taylor E. F., Wheeler J. A., *Spacetime Physics*, Freeman Ed., S Francisco, 1963
- [13] Rindler W., (1966), *Amer. Jour. Phys.*, **34**, 1174-1178
- [14] Desloge E. A., Philpott R. J., (1987), *Amer. Jour. Phys.*, **55**, 252-261